Learning to Measure Our Impact: Focus on Mathematics

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TODAY'S AGENDA

1. Background on our work
2. Mathematical habits of mind and teachers
3. Paper and pencil assessment
4. Wrap-up

FOCUS ON MATHEMATICS

Focus on Mathematics (FoM) is…

- a Targeted MSP funded by the NSF since 2003,
- a unique partnership of teachers, students, administrators, mathematicians, and mathematics educators—all committed to increasing student achievement, and
- a collaboration between:
  - Boston University and Education Development Center, Inc.
  - University of Massachusetts Lowell
  - Evaluators: Lesley University (PERG)
  - **Partner school districts:** Arlington, Chelsea, Lawrence, Waltham, Watertown; Cambridge, Framingham

FoM goals

- Provide teachers with:
  - coherent, content-focused professional development,
  - sustained immersion in mathematics,
- Develop mathematically expert teachers who share their knowledge with teachers and students,
- Build a mathematical learning community in which teachers and mathematicians work together, doing mathematics, and
- Improve student achievement.

Effective mathematics teaching

"An effective mathematics curriculum is one that

- is rich with mathematics,
- pays attention to students, and
- finds a way of connecting the two."

— W. J. McCallum, in testimony to the Nat'l Academy of Sciences

But how we implement this depends on our beliefs

- about mathematics, and
- about students and how students learn.

THE NATURE OF MATHEMATICS

- Mathematics is natural
  - The empirical nature of mathematics
  - People do mathematics naturally
- Mathematics exists independent of us
  - We can perform experiments
  - We can test ideas and decide for ourselves
- Experience precedes formality
  - "Meaning" is determined by experience
  - Definitions and theorems are capstones
  - Language is a tool for coming to terms with experience
- Mathematics is the study of structure
  - Operations, order
  - Shape
  - Continuity
  - Transformation
- **Mathematics is the art of figuring things out**

Experience first:

It has been observed in every human activity experience comes first, and as this experience grows the need for communication motivates the development of language. Sadly enough, in our classroom practice we place language first and experience second. We worry about what we should say in order to help the student "understand." By this we mean to provide the effect of experience through the use of suitably chosen words. Not unexpectedly, the effect is at best a very pale image of the real thing.

— Arnold Ross

The FoM Partnership has created

- School based study groups
- Seminars, colloquia, and summer institutes
- New graduate degrees and avenues for teacher leadership
- Online problem solving courses for teachers
- Student mathematics fairs (over 10,000 students participating)
- Research collaboratives
- Case studies of participating teachers
- Research study to measure teachers' MHoM (in FoM-II)

Impact on teachers

FoM teachers have reported[1]:

- Deeper knowledge of mathematics
- Changes in beliefs about the nature of mathematics and how students learn mathematics
- Renewed passion for mathematics
- Changes in instructional practice, for example
  - use of precise language
  - connecting mathematical ideas
  - mining student ideas and approaches

[1] Baldassari, C., Lee, S., & Torres, R. T. (2009). *The case of a high school mathematics teacher.* Retrieved from http://focusonmath.org/FOM/PERG

MATHEMATICAL HABITS OF MIND

Our approach: Over the years, FoM has offered a variety of opportunities designed to help teachers develop the mathematical habits of mind that are central to the discipline of mathematics.

We define mathematical habits of mind (MHoM) to be:

> *the specialized ways of approaching mathematical problems and thinking about mathematical concepts that resemble the ways employed by mathematicians.*

THE PRACTICE OF MATHEMATICS

The widespread utility and effectiveness of mathematics come not just from mastering specific skills, topics, and techniques, but more importantly, from developing the ways of thinking—the habits of mind—used to create the results.

— Cuoco, Goldenberg, & Mark

THE NOTION OF MATHEMATICAL PRACTICE

Designers of curricula, assessments, and professional development should all attend to the need to connect the mathematical practices to mathematical content in mathematics instruction.

— CCSS, 2010

TEACHERS AND MHOM

- Our FoM work has convinced us of the importance of MHoM for students and for *teachers* of mathematics, particularly at the secondary level.
- These habits foster the development and use of general purpose tools that make connections among various topics and techniques of secondary school mathematics content.
- They can bring parsimony, focus, and coherence to teachers' mathematical thinking and, in turn, to their work with students.

Example: Factoring

From a published text (2010)

To factor a trinomial of the form $ax^2 + bx + c$ where $a > 0$, follow these steps:

A	B	C
F	H	D
G	I	E

1. Identify the values of a, b, c. Put a in Box A and c in Box B. Put the product of a and c in Box C.
2. List the factors of the number from Box C and identify the pair whose sum is b. Put the two factors you find in Box D and E.
3. Find the greatest common factor of Boxes A [sic] and E and put it in box G.

Example: Factoring

From a published text (2010)

A	B	C
F	H	D
G	I	E

4. In Box F, place the number you multiply by Box G to get Box A.
5. In Box H, place the number you multiply by Box F to get Box D.
6. In Box I, place the number you multiply by Box G to get Box E.

Solution: The binomial factors whose product gives the trinomial are

$$(Fx + I)(Gx + H).$$

USING THE STRUCTURE OF EXPRESSIONS

Factoring monic quadratics (i.e., “sum-product” problems):

$$x^2 + 14x + 48$$

Using the structure of expressions

Factoring monic quadratics (i.e., “sum-product” problems):

$$x^2 + 14x + 48$$

We have

$$(x+a)(x+b) = x^2 + (a+b)x + ab$$

So, find two numbers whose sum is 14 and product is 48.

Using the structure of expressions

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$$x^2 + 14x + 48$$

We have

$$(x+a)(x+b) = x^2 + (a+b)x + ab$$

So, find two numbers whose sum is 14 and product is 48.

$$(x+6)(x+8)$$

USING THE STRUCTURE OF EXPRESSIONS

What about this one?

$$49x^2 + 35x + 6.$$

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Here's where seeking and exploiting structure can help

$$49x^2 + 35x + 6 = (7x)^2 + 5(7x) + 6$$

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$$\begin{aligned} 49x^2 + 35x + 6 &= (7x)^2 + 5(7x) + 6 \\ &= \clubsuit^2 + 5\clubsuit + 6 \end{aligned}$$

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$$\begin{aligned} 49x^2 + 35x + 6 &= (7x)^2 + 5(7x) + 6 \\ &= \clubsuit^2 + 5\clubsuit + 6 \\ &= (\clubsuit + 2)(\clubsuit + 3) \end{aligned}$$

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USING THE STRUCTURE OF EXPRESSIONS

What about $6x^2 + 31x + 35$? Unfortunately, 6 isn't a square. But if it's not a square, let's make it a square.

$$6(6x^2 + 31x + 35) = (6x)^2 + 31(6x) + 210$$

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USING THE STRUCTURE OF EXPRESSIONS

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USING THE STRUCTURE OF EXPRESSIONS

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&= (6x + 10)(6x + 21) \\
&= 2(3x + 5) \cdot 3(2x + 7) \\
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\end{aligned}$$

Therefore, $6x^2 + 31x + 35 = (3x + 5)(2x + 7)$.

INITIAL MOTIVATION FOR RESEARCH

- Through our FoM work, we've seen that MHoM is indeed a collection of habits teachers can acquire, rather than some static you-have-it-or-you-don't way of thinking.
- And teachers reported to us that developing these habits has had a tremendous effect on their teaching.
- We recognized the need for scientific-based evidence to establish that these teachers have indeed learned MHoM and that these habits have had a positive impact on their teaching practice.
- The instruments to measure these habits did not yet exist.

Our research

- *Assessing Secondary Teachers' Algebraic Habits of Mind* is an NSF DRK-12 collaborative project funded in 2012 aimed at developing instruments to assess secondary teachers' MHoM.
- This work fits into a larger research agenda with the goal of understanding the connections between secondary teachers' MKT and their students' understanding and achievement.
- Our work, which started in FoM-II, is centered on the following research question:

> *What are the mathematical habits of mind that secondary teachers use, how do they use them, and how can we measure them?*

INSTRUMENTS FOR CONDUCTING RESEARCH

To investigate our research question, we've been developing:

- Detailed definition of MHoM, based on existing literature, our own experiences as mathematicians, and classroom observations.
- A paper and pencil (P&P) assessment that measures how teachers engage MHoM when doing mathematics for themselves.
- An observation protocol measuring the nature and degree of teachers' use of MHoM in their classroom work.

Important remark: We've seen the need for both instruments, and also the value of developing all three components together.

What we aren't studying

There are many aspects of teaching that we value but we are *not* studying right now. For example:

- Teachers' dispositions (at least not directly)
- Teachers' beliefs
- Classroom discourse

What we aren't creating

We are *not* creating instruments that we anticipate can say much about an individual teacher. **Our goal is to create tools for research.**

Assessment development

- Over the last couple of years, several rounds of design, pilot-test, data analysis, and revision of the instrument.
- Inter-rater reliability testing for some items (κ up to 0.906) and concurrent rubrics refinement.
- Problem-solving think-aloud with teachers, educators, and mathematicians. (Do their interpretations match our objectives?)
- Reliability testing has yielded Cronbach's alpha value of 0.86 and Guttman Lambda 6 value of 0.89.

MHoM: Structure and language

Our current focus is on two categories of MHoM:

- Seeking, Using, and Describing Mathematical Structure (SUDS)
  - SUDS1. Discovering structure that is not apparent at first.
  - SUDS2. Making use of structure to solve problems.
- Using Mathematical Language (LANG)
  - LANG1. Using language to acquire experience, clarity, and understanding.
  - LANG2. Exercising appropriate "mathematical hygiene" (i.e., using language precisely).

Note: Eventually, we will investigate other habits, e.g., "Performing purposeful experiments" and "Applying mathematical reasoning."

P&P assessment: Key features

Distinguishing features of the P&P assessment:

- **It measures how secondary teachers use MHoM in their own doing of mathematics, in familiar contexts.**
- Content is from secondary mathematics—i.e., mathematical problems that most teachers have the requisite knowledge to solve, or at least begin to solve.
- **We're interested in their approach, as opposed to whether or not they can arrive at a solution.**
- Our items are drawn from multiple sources, including our classroom observation work.

Maximum Value

Sample Item:

Find the maximum value of the function $f(x) = 11 - (3x - 4)^2$.

- Though most teachers obtained the same (correct) answer, there were vast variations in their approaches.
- These various approaches came in "clumps," as our advisors (assessment experts) and research literature[2] told us to expect.
- Using these responses, we developed a rubric that allows us to code how each teacher solved the problem.

[2] Erickson; Glasser & Strauss; Guba; Maxwell

CONNECTION TO CCSS MATHEMATICAL PRACTICES

7. Look for and make use of structure:

   They can see $5 - 3(x - y)^2$ *as* 5 *minus a positive number times a square and use that to realize that its value cannot be more than* 5 *for any real numbers* x *and* y.

Rubric code: SQUR

(SQUR) Since $(3x-4)^2$ represents the *square* of some number, it is always ≥ 0. Thus in the function $f(x) = 11 - (3x-4)^2$, we are always subtracting a non-negative number from 11. To maximize $f(x)$, we need $(3x-4)^2 = 0$ so that the maximum value is 11.

Sample solution:

$f(x) = 11-(3x-4)^2$. Anything squared is ≥ 0.

Therefore, 11 − (stuff squared) must be ≤ 11. So 11 is the max.

QUICK MATHEMATICAL NOTE

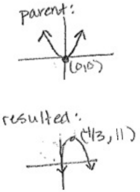
The reasoning described in SQR depends on the fact that x can be chosen so that $(3x-4)^2 = 0$. In many cases, we had no way of knowing whether the teachers actually noticed this detail.

RUBRIC CODE: TRNS

(TRNS) Uses transformation/shifting of the graph of $y = f(x)$. The graph is a downward facing parabola whose vertex is shifted up by 11 units (and to the right by $4/3$ units) from the origin. Thus the maximum value of $f(x)$ must be 11.

Sample solution:

Parent function is $f(x) = x^2$.
$f(x) = 11-(3x-4)^2$ is a matter of translations from the parent function.
$f(x) = -(3x-4)^2 + 11$ so it has shifted up 11 units $(+11)$, is flipped about the x-axis $(-)$, and shifted right 4/3 units $(3x-4)$.
Maximum value point is at $(\frac{4}{3}, 11)$.



RUBRIC CODE: SYMM

(SYMM) Expands $f(x)$ into $f(x) = -9x^2 + 24x - 5$. Finds the *axis of symmetry* using the formula $x = -b/(2a) = -24/(2(-9)) = 4/3$. Evaluates $f(4/3) = 11$ to obtain the maximum value.

Sample solution:

$$f(x) = 11 - (3x-4)^2$$
$$= 11 - (9x^2 - 24x + 16)$$
$$= 11 - 9x^2 + 24x - 16$$
$$= -9x^2 + 24x - 5$$

$$f\left(\tfrac{4}{3}\right) = 11 - \left(3\left(\tfrac{4}{3}\right) - 4\right)^2$$
$$= 11 - (4-4)^2$$
$$= \boxed{11}$$

x-coord. of vertex:

$$\frac{-b}{2a} = \frac{-24}{2(-9)} = \frac{-24}{-18} = \frac{4}{3}$$

max value is 11.

RUBRIC CODE: CALC

(CALC) Uses *calculus*. Sets $f'(x) = 0$ to find the critical pt. $x = 4/3$. Then evaluates $f(4/3) = 11$ to obtain the maximum value of $f(x)$. Some may also find $f''(x) = -18 < 0$ to conclude that f has a maximum at the critical point.

Sample solution:

$$f(x) = 11 - (9x^2 - 12x - 12x + 16) = -9x^2 + 24x - 5$$

$$f'(x) = -18x + 24$$

$$-18x + 24 = 0 \quad x = \frac{4}{3}$$

when $x = \frac{4}{3}$

$$f\left(\tfrac{4}{3}\right) = 11 - \left(3\left(\tfrac{4}{3}\right) - 4\right)^2 = 11$$

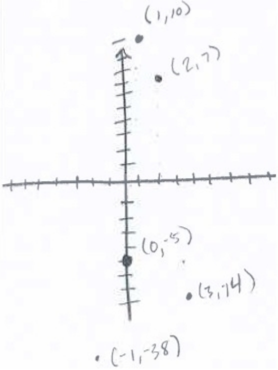
RUBRIC CODE: PNTS

(PNTS) Obtains *points* (x, y) satisfying the equation $y = f(x)$ by substituting in different values of x into $f(x)$. Uses these points to draw a rough sketch of the graph (although this is not required).

Sample solution:

x	$11-(3x-4)^2$	y
-2	$11-(-6-4)^2$	-89
-1	$11-(-3-4)^2$	-38
0	$11-(-4)^2$	-5
1	$11-(3-4)^2$	10
2	$11-(6-4)^2$	7
3	$11-(9-4)^2$	-14
4	$11-(12-4)^2$	-53

max will be somewhere in this range.



Intent of the Item

This item is intended to measure the SUDS2:

Making use of structure to solve problems.

- An advantage of $f(x) = 11 - (3x - 4)^2$ is that it immediately reveals the maximum value of f.
- Solution coded as SQUR or TRNS demonstrates purposeful use of the structure of this expression.
- Approaches SYMM, CALC, and PNTS are not addressing the underlying algebraic structure of the expression.

Some data from the 2012 pilot tests

Of the 59 teachers who took the assessment:

SQUR	TRNS	SYMM	CALC	PNTS	NEVD	OTHR
13	10	10	8	8	14	2

Remarks:

- There were some overlap in the coding, i.e., teachers who solved the problem in multiple ways.
- Again, the majority of teachers obtained the correct answer.

Sum of Squares

Sample Item:

Two numbers *a* and *b* have a sum of 3 and a product of 3. What is the sum of their squares?

Note: If you cannot complete the necessary calculations, describe how you would finish this problem.

Discussion Questions:

- Is the mathematics content *relevant* to secondary teaching and *accessible* to secondary teachers?
- Which mathematical habit(s) does this item measure?

SAMPLE CODE: IDNT

(IDNT) Uses the identity $(a+b)^2 = a^2 + 2ab + b^2$ by substituting the given values $a + b = 3$ and $ab = 3$ and then solving for $a^2 + b^2$.

Sample solution:

$(a+b)(a+b) = a^2 + 2ab + b^2$

$3 \cdot 3$ ↑ 6

$9 = a^2 + b^2 + 6$

$a^2 + b^2 = 3$

SAMPLE CODE: ELIM

(ELIM) Solves the system of equations $a + b = 3$ and $ab = 3$ by eliminating one of the variables. Then computes $a^2 + b^2$ upon solving the system and finding the values of a and b.

Sample solution:

$A + B = 3$

$(A)(B) = 3$

Then $B = \frac{3}{A}$,

$A + \frac{3}{A} = 3$,

then

$A^2 + 3 = 3A$,

$$A^2 - 3A + 3 = 0 \Rightarrow \frac{3 \pm \sqrt{9 - 4(1)(3)}}{2} = \frac{3 \pm \sqrt{-3}}{2}$$

Since A and B are arbitrary, let $A = \frac{3 + i\sqrt{3}}{2}$ and $B = \frac{3 - i\sqrt{3}}{2}$

Note: $A + B = 3$ & $(A)(B) = 3$ holds.

Then $A^2 + B^2 = \left(\frac{3 + i\sqrt{3}}{2}\right)^2 + \left(\frac{3 - i\sqrt{3}}{2}\right)^2$

$$\frac{9 + 6i\sqrt{3} - 3}{4} + \frac{9 - 6i\sqrt{3} - 3}{4} = \frac{6}{4} + \frac{6}{4} = \boxed{3}$$

Next steps on the P&P

What's next?

- Upcoming rounds of smaller field tests (we just can't stop!)
- More validity and reliability testing of the assessment
- Larger field test 2014–2015: 200 teachers
- Student data collection (PARCC assessments)

Thank you

- Thank you for your participation and feedback!
- If you have further feedback and/or questions, email us at:
  - Sarah Sword (ssword@edc.org)
  - Ryota Matsuura (matsuura@stolaf.edu)
- And be sure to visit our website: mhomresearch.edc.org

POWER OF NUMBER b

Sample Item:

Imagine you have a number b, not equal to 1, such that $b^7 = 1$.
Find the smallest whole number n (i.e., $n \geq 0$) such that $b^{374} = b^n$.